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# PLANE SYMMETRY GROUPS

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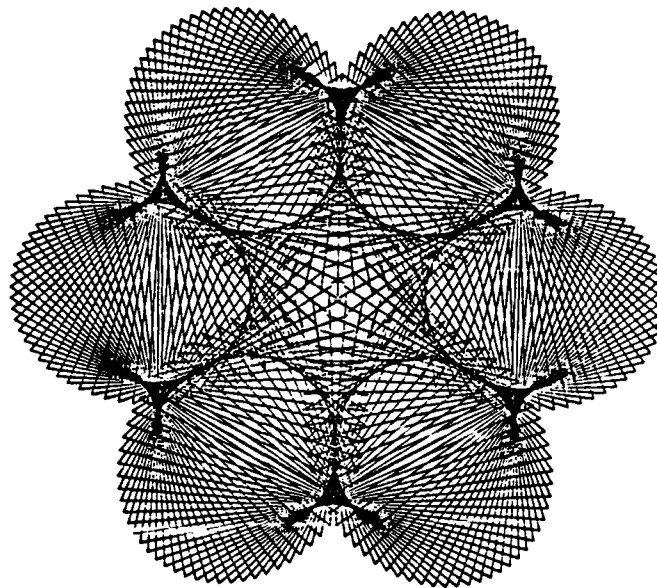
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<p>Plane symmetry groups are a classification system for describing the symmetry of two-dimensional figures and patterns. This paper presents the basic mathematical theory behind plane symmetry groups. This theory is then applied in classifying the symmetry of bounded figures, frieze patterns and wall paper patterns. Recently developed algorithms are included to help analyze complex designs. Over 100 examples are presented in the text to clarify concepts and to illustrate the various symmetry types. (500)</p>				
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PLANE SYMMETRY GROUPS

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June 1988

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"That whenever any style of ornament commands universal admiration, it will always be found to be in accordance with the laws which regulate the distribution of form in nature."

- Owen Jones, "The Grammar of Ornament", 1856

"I often wondered at my own mania of making periodic drawings. Once I asked a friend of mine, a psychologist, about the reason of my being so fascinated by them, but his answer: that I must be driven by a primitive, prototypical instinct, does not explain anything."

- M. C. Escher, Preface to "Fantasy and Symmetry", 1965

## INTRODUCTION

Few applications in the mathematical field of group theory are as readily understandable as the useful beauty of plane symmetry groups. Plane symmetry groups are a classification system for describing the symmetry of two-dimensional figures and patterns. Here mathematics truly comes alive, as abstract symbols used to describe symmetry can be immediately visualized as the rotation of a snowflake, the pattern on a vase, or the ceramic tile of an ornate kitchen floor. The goal of this paper: to introduce the basic theory behind plane symmetry groups, and to present some simple algorithms one may use to analyze complex designs.

To make the theory accessible to as broad an audience as possible, the mathematics has been deliberately downplayed. Simple proofs that would be of interest to the undergraduate math major can be found in the appendix. More difficult or tedious proofs are relegated to the bibliography. Since mosaics and patterns are visual creations, examples and illustrations are used frequently in the text. The prerequisites to understanding the contents of this report are modest. The authors assume that the reader is comfortable with the basic notation and theory for sets, functions, and the composition of functions.

As we progress, several fascinating topics will branch off from our main theme. As alluring, beautiful, and important as these topics may be, we will not take the time to investigate them here. Rather, such topics will be referred to the bibliography, affording the ambitious reader the opportunity to progress further on his own.

This paper is the joint effort of Major David Jensen and Cadet Firstclass Gary Harvey. It represents partial fulfillment of Math 499 (Independent Study) course requirements for Cadet Harvey, a senior math major at the Air Force Academy. The material for this technical report comes largely from a two-hour talk Major Jensen developed to introduce plane symmetry groups using the brilliant work of the Dutch artist, M. C. Escher.



## Chapter 1

### BASIC THEORY

A binary operation is a function that takes two elements in a nonempty set  $G$  and assigns to them a unique element also in the set  $G$ . Using standard function notation, if  $*$  is a binary operation on  $G$  we write  $*$  :  $G \times G \rightarrow G$ . Familiar examples of binary operations abound, with one of the easiest being the real numbers under normal addition ( $+$  :  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ).

With this notion of a binary operation we can define a group--one of the most important algebraic structures in the world of mathematics.

#### Definition 1.1 Group $(G,*)$

A group  $(G,*)$  is any nonempty set  $G$  together with a binary operation  $*$  :  $G \times G \rightarrow G$  that satisfies the following three properties:

Associative Property:  $a * (b * c) = (a * b) * c$ , for all  $a, b$ , and  $c \in G$ .

Identity Property: There exists  $e \in G$  such that  $a * e = e * a = a$ , for all  $a \in G$ . We call  $e$  the identity element of the group.

Inverse Property: For every  $a \in G$ , there exists  $b \in G$ , such that  $a * b = b * a = e$ .

We call  $b$  the inverse of  $a$  and write  $b = a^{-1}$

We say  $(H,*)$  is a subgroup of a given group  $(G,*)$  when  $H$  is a nonempty subset of  $G$ , and  $(H,*)$  is itself a group. Also, a group  $(G,*)$  is called Abelian if all its elements commute, that is  $a * b = b * a$  for all  $a$  and  $b$  in  $G$ .

The definition of a group is important because the properties listed above give it enough structure to be useful, while at the same time the definition is not too restrictive. There are lots of groups. The real numbers under normal addition  $(R, +)$  form a group with identity  $e = 0$  and inverses of the form  $a^{-1} = -a$ . Rather than  $R$ , we could have just as easily chosen the rational numbers  $Q$  or the integers  $I$  and formed the groups  $(Q, +)$  and  $(I, +)$ . For another example, the real numbers (excluding zero) under normal multiplication,  $(R - \{0\}, \cdot)$ , form a group with identity  $e = 1$  and inverses given by  $a^{-1} = 1/a$ .

It is also easy to define groups of matrices or groups of functions under various binary operations. In particular, consider  $R_2$ , the set of all points in the plane, and let  $G$  be the set of all one-to-one functions from  $R_2$  onto  $R_2$ . Then consider  $(G, \circ)$  where  $\circ$  represents composition of functions. First note that all one-to-one, onto functions from  $R_2$  to  $R_2$  are invertible. Therefore, elements in  $G$  have inverses. Moreover,  $\circ$  is a binary operation on  $G$  since the composition of two invertible functions is again an invertible function. The other two properties needed to establish that  $(G, \circ)$  is a group follow readily from the fact that all invertible functions from  $R_2$  to  $R_2$  are associative under composition, and the identity function  $i : R_2 \rightarrow R_2$  defined by  $i(p) = p$  for all  $p \in R_2$  is the logical choice for the group identity element.

When investigating the symmetry properties of plane figures and patterns, the group  $(G, \circ)$  is too large to be very useful. Our first real progress in applying group theory to questions of symmetry comes when we consider a special subset  $H$  of  $G$ . We let  $H$  be precisely the invertible maps from  $R_2$  to  $R_2$  that also preserve distance between points. Using the usual notation of vertical lines

for distance, we have  $H = \{\alpha \in G : |p - q| = |\alpha(p) - \alpha(q)| \text{ for all } p, q \in R_2\}$ , where  $|p - q|$  is the straight line distance from point  $p$  to point  $q$  and  $|\alpha(p) - \alpha(q)|$  is the straight line distance from point  $\alpha(p)$  to point  $\alpha(q)$ . It is easy to show that  $H$  is a subgroup of  $G$  (see appendix A), and we call the elements of  $H$  the motions, or isometries, of the plane. Moreover, it can be shown that there are only four types of motions possible [1]:

### 1. Translation

A mapping  $\alpha$  that sends all points in  $R_2$  the same distance  $d$  in the same direction  $\theta$ . To illustrate, consider  $p_i, q_i \in R_2$  with  $\alpha(p_i) = q_i$ ,  $i = 1, 2, 3$ :

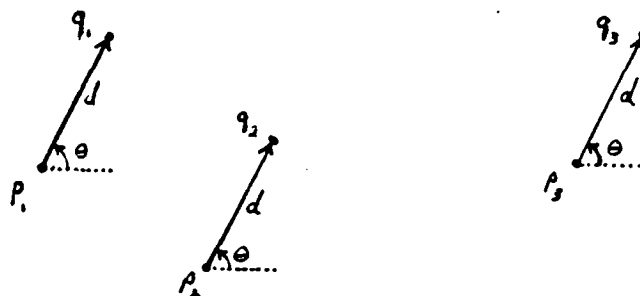


Figure 1.1

### 2. Rotation

A mapping  $\alpha$  obtained by rotating the plane clockwise a fixed amount  $\phi$  about a fixed point  $p$ . To illustrate, consider  $p_1, q_1 \in R_2$  with  $\alpha(p_1) = q_1$ :

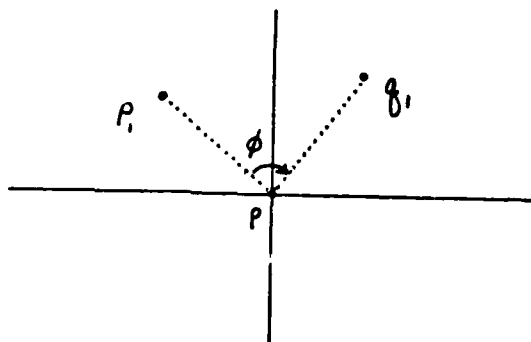


Figure 1.2

### 3. Mirror

A mapping  $\alpha$  obtained by reflecting the plane through a fixed line  $L$  (that is, a mapping that sends each point  $p$  to a point  $q$  such that  $L$  is the perpendicular bisector of the straight line between  $p$  and  $q$ ). To illustrate consider  $p_i, q_i \in R_2$  with  $\alpha(p_i) = q_i, i = 1, 2, 3, 4$ :

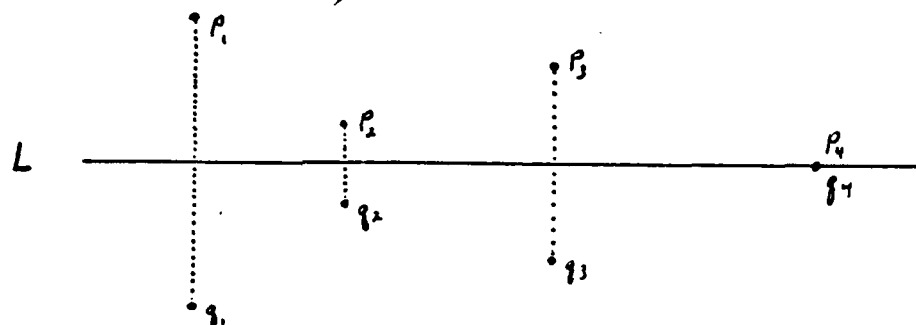


Figure 1.3

### 4. Glide

A mapping  $\alpha$  composed of a translation in the direction of a fixed line  $L$ , followed by a mirror through  $L$ . To illustrate let  $p_i, q_i \in R_2$  with  $\alpha(p_i) = q_i, i = 1, 2$ :

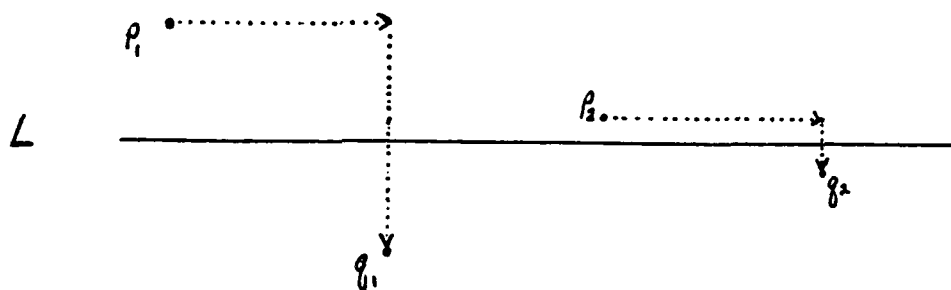


Figure 1.4

Having defined the subgroup  $(H, \circ)$  of  $(G, \circ)$  we are nearing our goal of being able to use group theory to analyze the symmetry of figures and designs in the plane. The problem is that  $(H, \circ)$ , as a set, is still too large. The next

definition overcomes this problem by restricting  $(H, \circ)$  in a very natural way, leaving us with precisely the motions we need to describe the symmetry of a given figure.

Definition 1.2      Symmetry Group of  $T$

Let  $T$  be any nonempty set of points in the plane,  $T \subseteq \mathbb{R}_2$ . Define a subset  $H_T$  of  $H$  by  $H_T = \{\alpha \in H : \alpha(T) = T\}$ . Here  $\alpha(T) = T$  denotes set invariance, that is  $\alpha(p) \in T$  for every  $p \in T$ . It can be shown (see appendix B) that  $H_T$  is itself a subgroup of  $G$ . We call  $H_T$  the symmetry group of  $T$ .

The way to view this definition of  $H_T$  is as follows:

- Start with a given figure in the plane. For a simple example, take a circle of radius  $r$  centered at the origin of the Cartesian coordinate system.
- Consider the points that make-up the figure to be  $T$ . Therefore, for our example, the set  $T$  is the locus of points satisfying  $x^2 + y^2 = r^2$ .

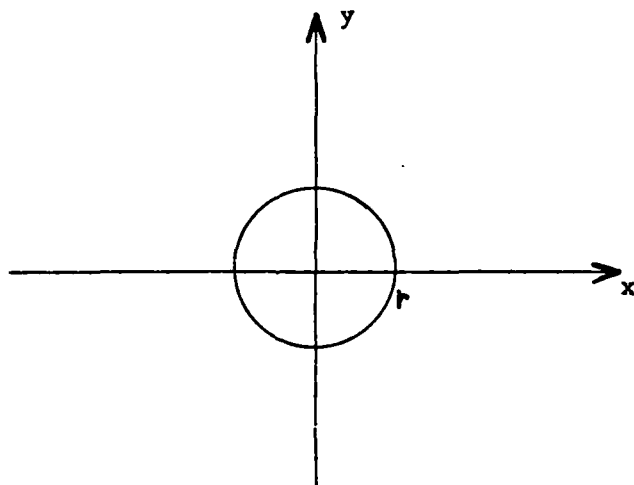


Figure 1.5

- Then  $H_T$  is exactly those translations, rotations, mirrors and glides that map  $T$  back onto itself. When  $T$  is the circle shown in Figure 1.5 (in fact, when  $T$  is any bounded figure) we will see in Chapter 2 that translations and glides cannot be elements of  $H_T$ . If we consider rotations, there are obviously an infinite number of possibilities, since any rotation about the origin will leave  $T$  invariant. In addition, any mirror through a line passing through the origin will also map  $T$  onto itself.

Example 2.1 Find the symmetry group of  $T$  where  $T$  contains only two distinct points, say  $T = \{p_1, p_2\}$ . Note first that the identity map  $i$  is in  $H_T$ . For if  $i(p) = p$  for every  $p$  in the plane, then certainly  $i(T) = T$ . To determine the other motions in  $H_T$ , let  $L_1$  be the straight line through points  $p_1$  and  $p_2$ , and let  $L_2$  be the perpendicular bisector of the line segment from  $p_1$  to  $p_2$ .

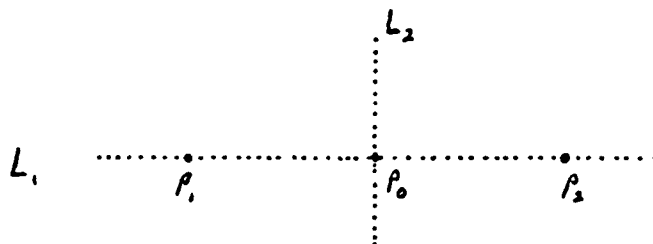


Figure 1.6

Using the definitions of a translation and a glide, it is easy to see that as in the case of the circle, translations and glides cannot be elements of  $H_T$ . The only rotation in  $H_T$  is the rotation of  $180^\circ$  about the point  $p_0$ . (We don't count the case where we pick a point, say  $p_1$ , and rotate everything  $360^\circ$  about that point. After all this just yields the identity map, which we have already acknowledged as being in  $H_T$ .) The only two mirrors possible are reflections through the lines  $L_1$  and  $L_2$ . Therefore,  $H_T$  contains exactly four elements: the identity map,  $180^\circ$  rotation about  $p_0$ , reflection through  $L_1$ , and reflection through  $L_2$ .

Example 2.2 Find the symmetry group of  $T$  where  $T$  is the set of points that make-up the footsteps depicted below [2]:

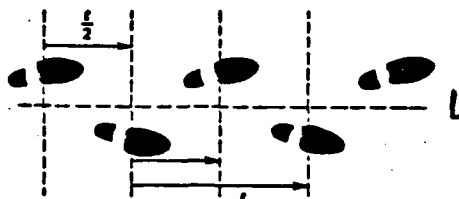


Figure 1.7

The footprints are assumed to continue infinitely to the right and to the left. There are no rotations or mirrors in  $H_T$ . However, this is the first example we have encountered where translations play a part. A translation of length  $t$  (or any integer multiple of  $t$ ) in the direction of  $L$  will map  $T$  onto itself. There is also a glide in this case consisting of a  $t/2$  translation (or any integer multiple of  $t/2$ ) in the direction of  $L$  followed by a reflection through  $L$ . Note that because different integer multiples of the period  $t$  ( $t/2$ ) give rise to different translations (glides),  $H_T$  has an infinite number of elements. A symmetry group that has an infinite number of elements is called an infinite symmetry group. Likewise, a finite symmetry group is one with only a finite number of elements.

A major goal of this paper was to introduce the basic theory behind plane symmetry groups. In this chapter we have accomplished that by developing the foundational idea of the symmetry group of a set  $T$ . In the next three chapters we will see how to use this idea to classify the symmetry of different figures and designs in the plane. Specifically, we will accomplish the following:

Chapter 2: Classify symmetry groups for plane bounded figures.

Chapter 3: Classify infinite symmetry groups for patterns that repeat themselves regularly in one dimension. (Frieze Groups)

Chapter 4: Classify infinite symmetry groups for patterns that repeat themselves regularly in two dimensions. (Wallpaper Groups)



## Chapter 2

### SYMMETRY GROUPS FOR PLANE BOUNDED FIGURES

A bounded figure in the plane is one which can be encompassed by a circle of finite radius. In this chapter we classify the types of symmetry groups, that is the sets of motions  $H_T$ , that are possible for plane bounded figures. The task is easier than it might first appear. Translations (and glides) cannot be motions in the symmetry group of a set  $T$  which represents a bounded figure. A simple proof of this fact can be found in Appendix C. Therefore, in dealing with plane bounded figures, we need only consider rotations and mirrors.

Consider the symmetry group of an equilateral triangle:

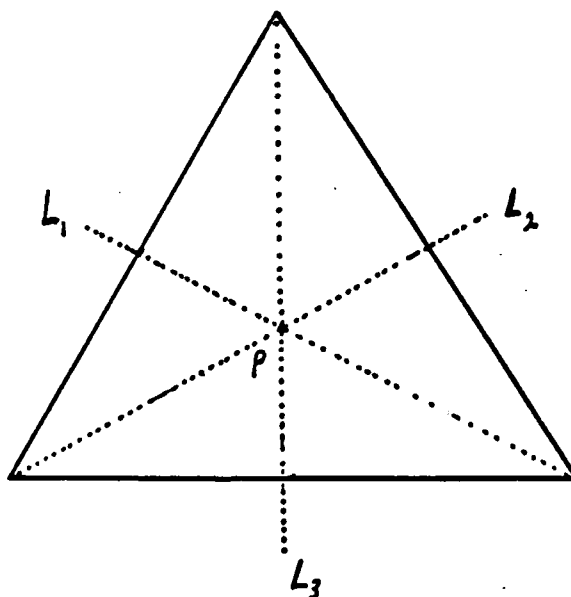


Figure 2.1

$H_T = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$  where

$\alpha_1$  = identity map

$\alpha_2$  = rotation  $120^\circ$  clockwise around  $p$

$\alpha_3$  = rotation  $240^\circ$  clockwise around  $p$

$\alpha_4$  = reflection through  $L_1$

$\alpha_5$  = reflection through  $L_2$

$\alpha_6$  = reflection through  $L_3$

We call this symmetry group a dihedral group and write  $H_T = D_3$ . The order of a group is simply the number of elements in the group. Therefore, the order of  $D_3$  is 6. A part of  $D_3$  that we are especially interested in is the subgroup  $C_3 = \{\alpha_1, \alpha_2, \alpha_3\}$ . Note that  $C_3$  can be generated by repeated compositions of the single rotation  $\alpha_2$ :

$$C_3 = \{\alpha_2, \alpha_3, \alpha_1\} = \{\alpha_2, \alpha_2 \circ \alpha_2, \alpha_2 \circ \alpha_2 \circ \alpha_2\} = \{\alpha_2, \alpha_2^2, \alpha_2^3\}$$

A group which is generated by a single element in the group is called a cyclic group. Therefore  $C_3$  is a cyclic subgroup of order 3.

From the development of  $D_3$ , the symmetry group of an equilateral triangle, it is easy to envision a similar development for the symmetry group of a square. We would obtain a dihedral group  $D_4$  with eight elements, (4 reflections and 4 rotations). Once again, the rotations would form a cyclic subgroup, in this case  $C_4$ . More generally the symmetry group of any regular  $n$ -sided polygon is  $D_n$  (with subgroup  $C_n$ ) where [2]:

$C_n$  is a cyclic group of order  $n$ , consisting of clockwise rotations through  $k \left(\frac{360^\circ}{n}\right)$ ,  $0 \leq k < n$ , around a fixed point  $p$ .

$D_n$  is a dihedral group of order  $2n$  and consists of  $C_n$  together with reflections through  $n$  axes that intersect at  $p$  and divide the plane into  $2n$  equal angular regions.

With these definitions of  $C_n$  and  $D_n$  we can now classify all possible finite symmetry groups for plane bounded figures. Specifically, we have the following powerful result:

Theorem 2.1 A finite symmetry group of a plane bounded figure must be either a cyclic group  $C_n$  or a dihedral group  $D_n$ .

An especially well-written proof of Theorem 2.1 is provided by Durbin [2]. We will not discuss the proof here except to note that the word "finite" is important. The circle we discussed in Chapter 1 is certainly a plane bounded figure, but it cannot be classified as either  $C_n$  or  $D_n$  for finite  $n$ . As we saw earlier, the symmetry group of a circle contains an infinite number of rotations and reflections. The symmetry groups for circular figures are a special case and are called continuous symmetry groups. They are often denoted  $C_\infty$ . Except for circular figures, all plane bounded figures have finite symmetry groups and Theorem 2.1 applies.

While  $C_n$  and  $D_n$  (and  $C_\infty$  for circular figures) classify the symmetry for bounded figures, we still need to address the more difficult unbounded case. That's the next challenge to be taken up in Chapters 3 and 4 where we will look at patterns (figures that repeat themselves at regular intervals in the plane). Before moving on however, let's consider some especially beautiful examples of plane bounded figures. The cardioids and roses that follow were derived from

Dr. Peter M. Mauer's recent work in computer graphics [7]. A special thanks to Lt Colonels William J. Riley and Robert L. James, Office of the Dean of the Faculty, U. S. Air Force Academy, Colorado, for programming Dr. Mauer's algorithm and actually generating the illustrations.

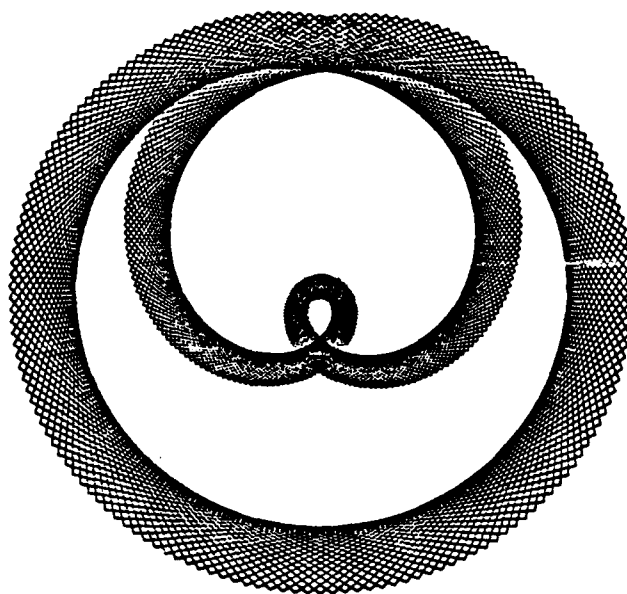


Figure 2.2

"Spiral of Archimedes"

Symmetry:  $D_1$  (Bipolar Symmetry)

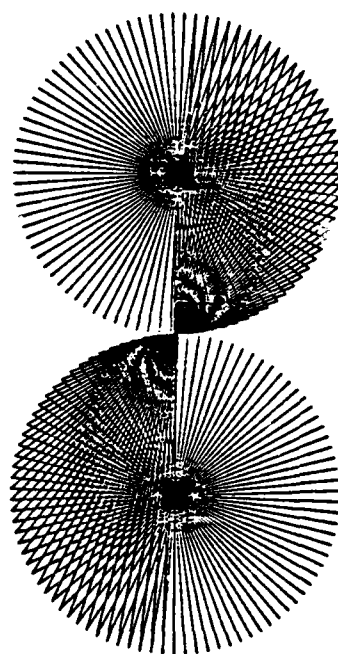


Figure 2.3

Symmetry:  $C_2$

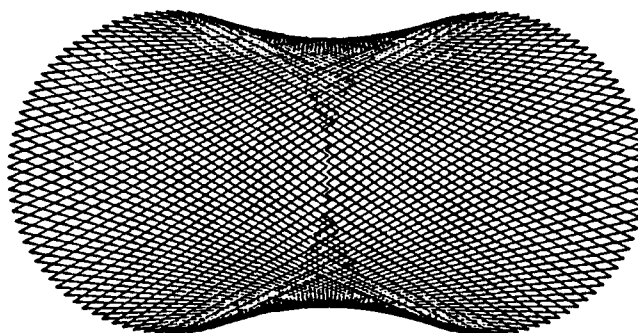


Figure 2.4

Symmetry:  $D_2$  (Klein 4-Group)

Labeling Figure 2.4 as  $D_2$  is an example of the liberty one might take in classifying the symmetry of a given design (note that a reflection  $\alpha$  through a vertical line through the center of 2.4 does not quite yield invariance  $\alpha(T) = T$ , since the two sides do not perfectly align). In fact, with every picture of a planar figure one should remember that perfect symmetry is a mathematical ideal--an ideal not fully realized by the stroke of any pen. Indeed, if just one molecule of ink is "misplaced" after a motion, an infinite number of points are not invariant. So when is there "enough" symmetry present to classify a given picture as having a certain symmetry type? The answer is subjective. In many contexts, the following consideration is useful: did the artist who created the design intend for the viewer to interpret it as having perfect symmetry?

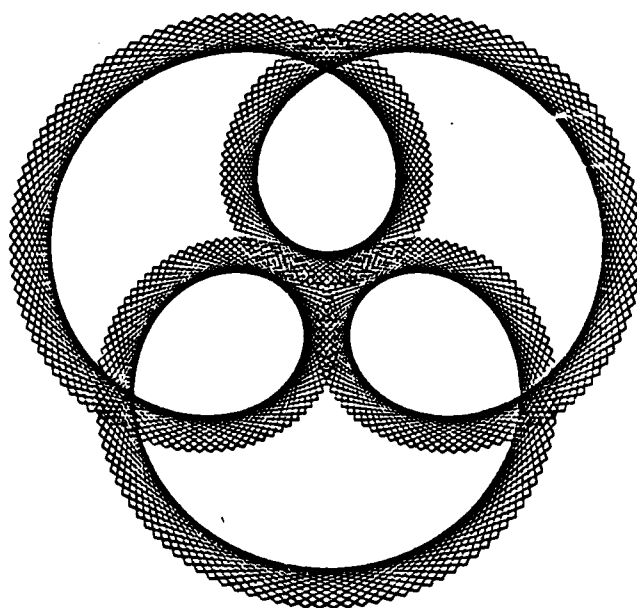


Figure 2.5

Symmetry:  $D_3$

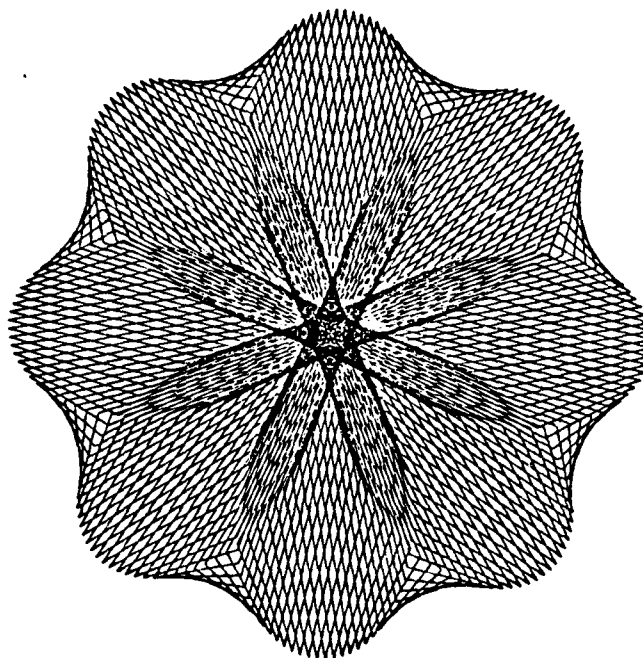


Figure 2.6  
Symmetry:  $D_8$

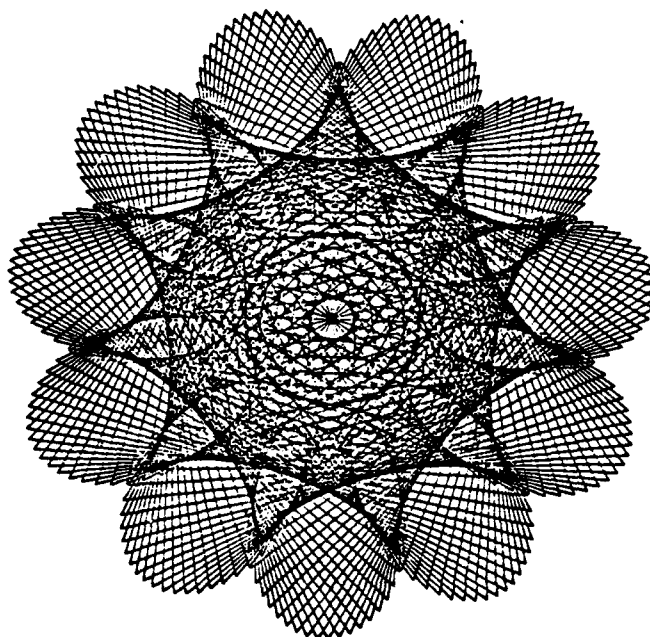


Figure 2.7  
Symmetry:  $D_{11}$



As a final example, the 26 letters of the alphabet represent 5 different symmetry types. One from each type is listed below. Can you classify the other 21 letters?

F -  $C_1$

M -  $D_1$

S -  $C_2$

H -  $D_2$

O -  $C_\infty$

Figure 2.8

## Chapter 3

### FRIEZE GROUPS

A frieze is any decorative strip or border that contains lettering, sculpture, pictures, etc. (In classical architecture, the frieze is that part of the entablature between the architrave and the cornice.) From our group symmetry point of view we are interested in those two-dimensional designs located in a frieze that repeat themselves at regular intervals. We assume these designs continue infinitely in both directions along a straight line. The footsteps we encountered in example 1.2 are a good example of a frieze pattern.

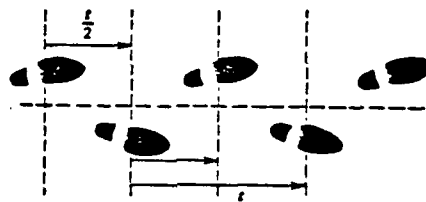


Figure 3.1

Like all frieze patterns, the symmetry group of the footprints is an infinite symmetry group. However, note that the footprints do have a minimum translation period, in this case  $t$ . The existence of a minimum translation period identifies

the pattern as having what is called a discrete symmetry group. This is not always the situation, as when we consider the stripe pattern depicted below:

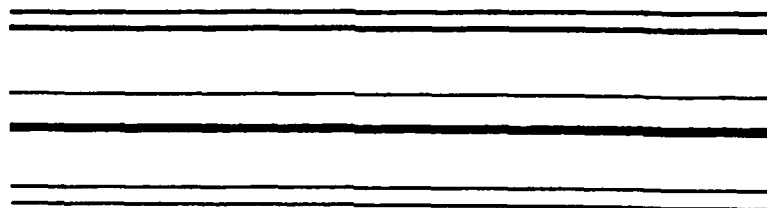


Figure 3.2

A stripe pattern has no minimum translation period and we say that its symmetry group is continuous (this is really the same idea we encountered with circles when dealing with bounded figures--there no minimum rotation existed). We will assume for the rest of this paper that we are dealing with only discrete symmetry groups.

The symmetry group of a frieze pattern is called a frieze group, and there are exactly seven types of frieze groups [6]. This classification is based on the fact that the only motions possible for a frieze pattern are:

- translations along a fixed line  $L$
- $180^\circ$  rotations about points on  $L$
- a horizontal mirror through  $L$
- vertical mirrors perpendicular to  $L$
- glides with respect to  $L$

Every frieze group must have translations, but it is the presence or absence of the other motions that defines the symmetry. The seven types of frieze groups are depicted in the following illustrations taken from John R. Durbin's book [2], "Modern Algebra: An Introduction."

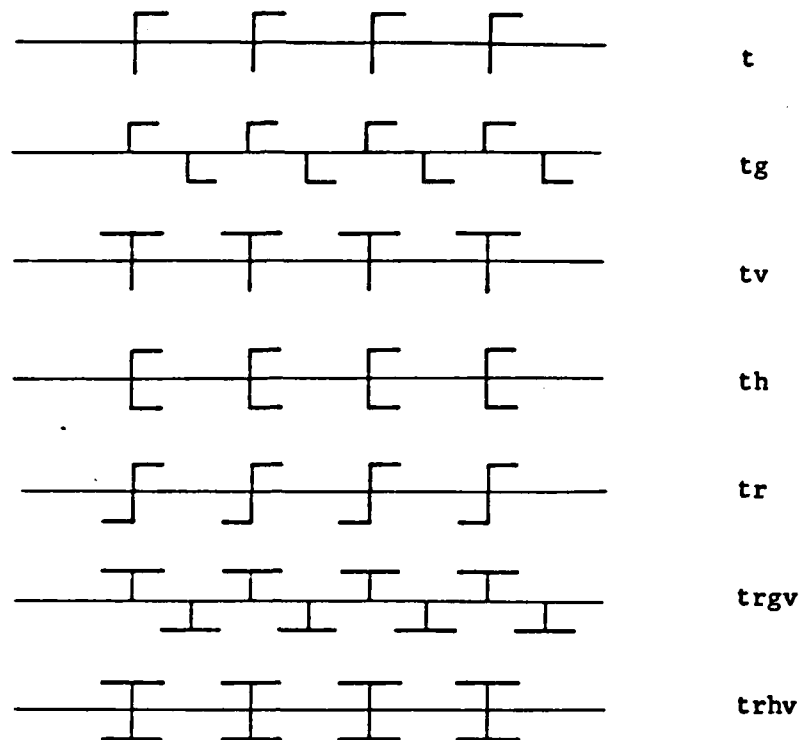


Figure 3.3

Key: t - translations  
 g - glides  
 v - vertical mirrors  
 h - horizontal mirror  
 r - rotations

Doctors Bruce Rose and Robert Stafford have recently created a simple algorithm to aid in classifying frieze patterns [8]. With slight modification, the algorithm is as follows:

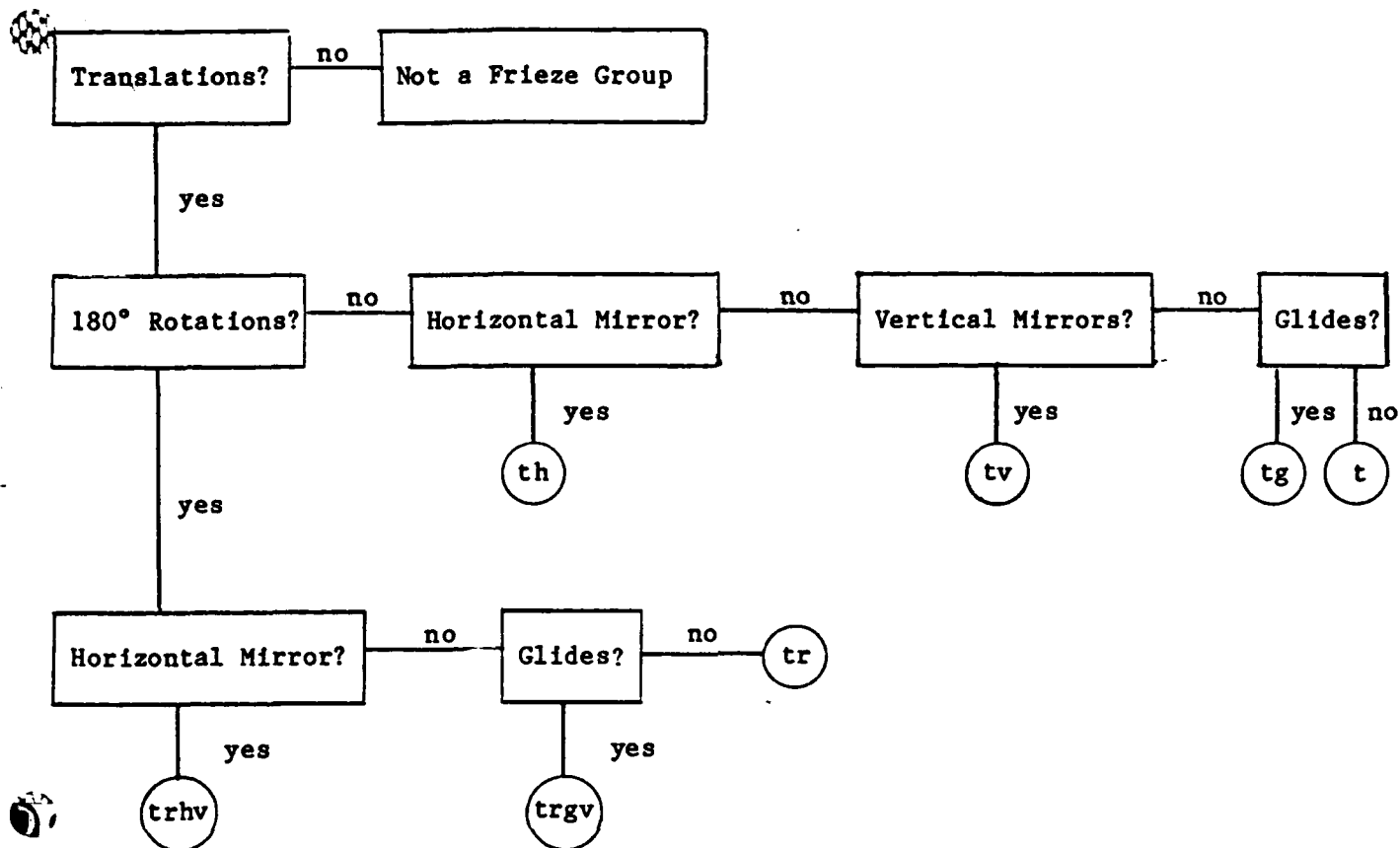


Figure 3.4

### Example 3.1

Classify the symmetry group  $H_T$  for the graph of  $f(x) = \sin x$ .

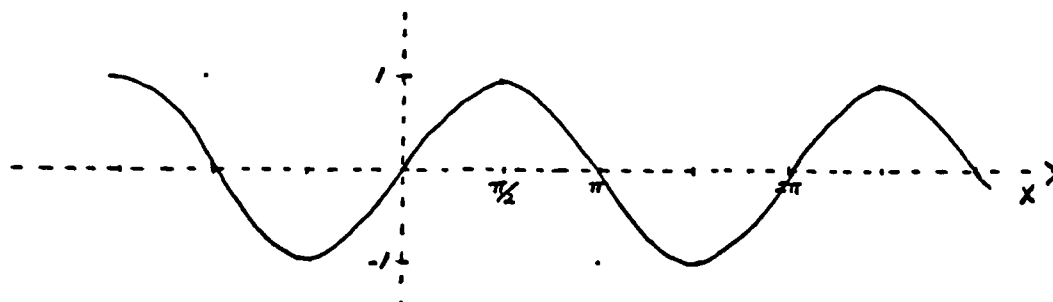


Figure 3.5

While translations along the x-axis (minimum period of  $2\pi$ ) are obvious, there are also  $180^\circ$  rotation points along the x-axis at  $\pm n\pi$ ,  $n$  an integer. Using the algorithm in Figure 3.4 we would next ask if  $H_T$  contains a horizontal mirror through the x-axis. It doesn't, but we do observe that a translation of  $\pi$  units along the x-axis followed by a reflection through that axis is a member of the symmetry group. Therefore, glides are elements of  $H_T$  and we conclude that the symmetry type is "trgv".

We conclude this chapter with seven illustrations taken from Owen Jones' classic "The Grammar of Ornament," first published in 1856 [4]. As an example of the impact color has on symmetry, notice that the coloring in the "trhv" illustration doubles the minimum translation period. Polychromatic Symmetry is a fascinating field and for those interested in learning about the impact of color on symmetry one of the most enjoyable places to start is Caroline MacGillavry's book [5], "Fantasy and Symmetry: The Periodic Drawings of M. C. Escher".

Key to Figure 3.6:

- t        - Medieval Stained Glass - Cathedral of Bourges
- tv       - Medieval Stained Glass - Cathedral of Bourges
- tr       - Persian Manuscript - British Museum
- trhv    - Persian Manuscript - British Museum
- th       - Medieval Stained Glass - Cathedral of Bourges
- tg       - Persian Manuscript - British Museum
- trgv    - Greek Vase - British Museum or the Louvre

t



tv



tr



trhv



th



tg



trgv

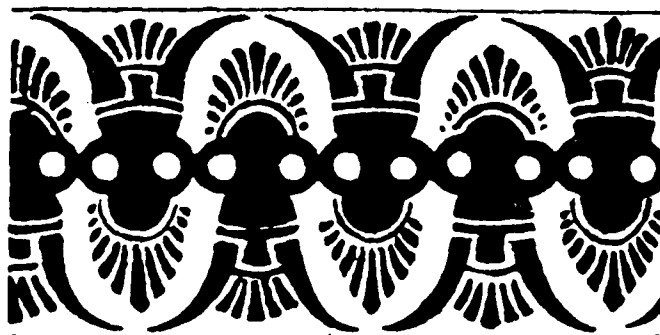


Figure 3.6

## Chapter 4

### WALLPAPER GROUPS

Wallpaper patterns are those patterns in the plane that repeat themselves at regular intervals in two non-parallel directions.

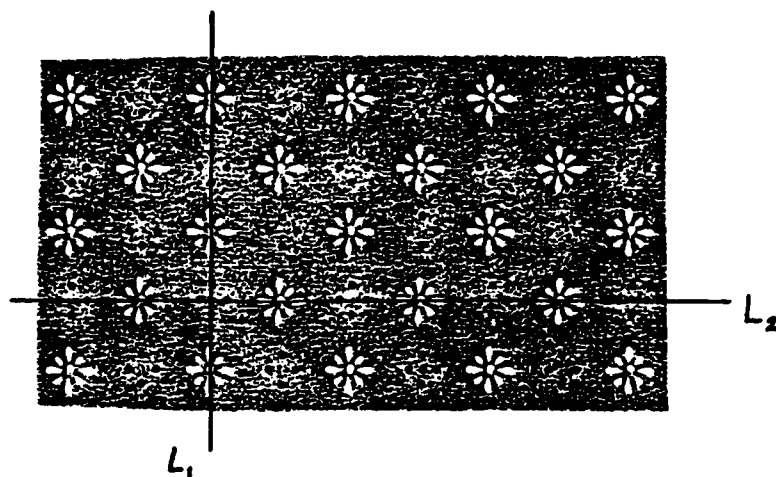


Figure 4.1

The above pattern is a reduced copy of an actual wallpaper sample. Notice that we have independent translations along the two lines  $L_1$  and  $L_2$ . We assume the wallpaper design repeats itself infinitely, filling the entire plane. We call the symmetry group of a wallpaper pattern a wallpaper group. As with a frieze group, a wallpaper group is an infinite symmetry group. The key to classifying wallpaper groups was unlocked in the 1890's by the Russian crystallographer E. S. Fedorov: there are only 17 types of wallpaper groups. We will not take up the proof of Fedorov's assertion except to say that at the heart of the proof lies one of the most elegant and useful tools found in any branch of mathematics, the Crystallographic Restriction. The Crystallographic Restriction tells us that the only nontrivial rotations possible in a wallpaper group are rotations of  $60^\circ$ ,  $90^\circ$ ,



120°, and 180°. The best informal discussions that explain the Crystallographic Restriction and why there are 17 types of wallpaper groups are given by Durbin [2] and Schattschneider [9]. For those with a hearty background in mathematics, a full group theory development is given by Schwarzenberger [11].

In theory, determining the symmetry type of a given wallpaper pattern should be easy and straightforward. In reality however, the symmetry type can often be devilishly obscure. Therein lies the challenge and fun. Fortunately, there are some marvelous aids to help us analyze complex designs. Schattschneider has compiled a useful table for classifying wallpaper patterns [9]. Virtually the same table has been put into algorithm form by Drs Rose and Stafford [8] and is reproduced from Durbin's book [2] in Figure 4.2.

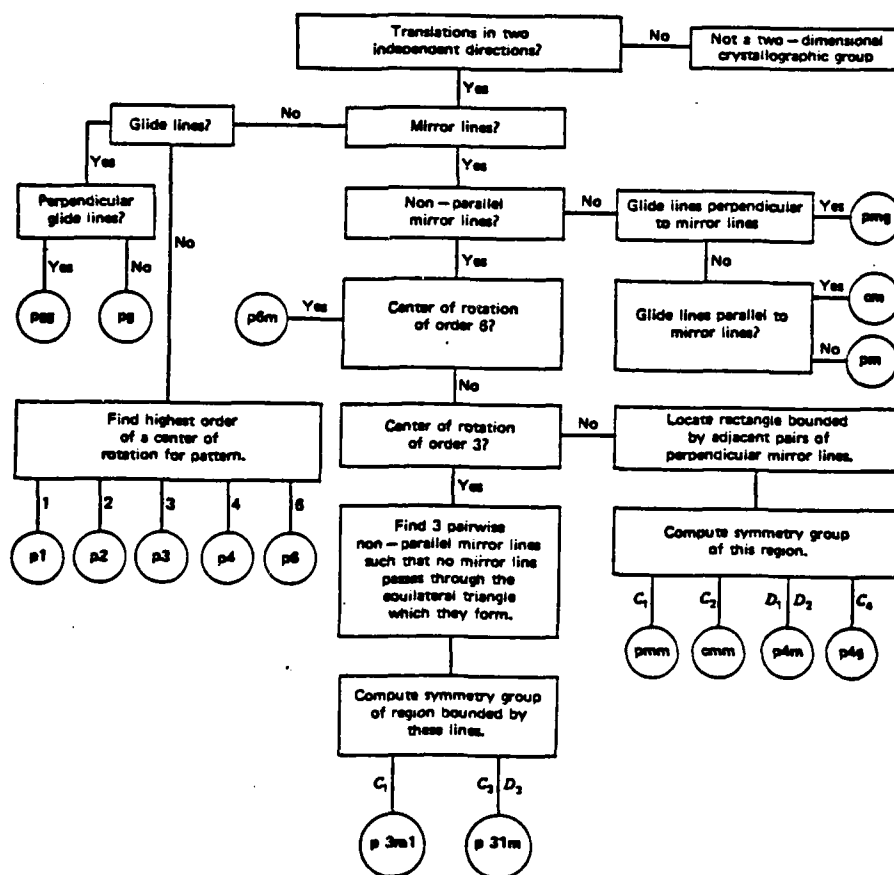


Figure 4.2

In Figure 4.2 the symbols used for the 17 types of wallpaper groups (also called two-dimensional crystallographic groups) are those most commonly accepted and come from a coding system designed by crystallographers. A full explanation of the symbols is given by Schattschneider [9].

Example 4.1 Classify the symmetry group  $H_T$  for the following illustration taken from Owen Jones' book [4], "The Grammar of Ornament."

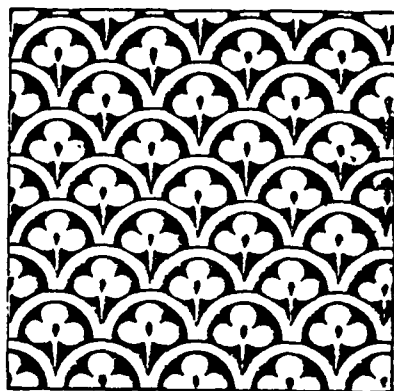


Figure 4.3

This pattern is easy to analyze using the algorithm in Figure 4.2. First, note that there are vertical mirror lines through the center of each leaf. By observation, these are the only mirror lines for this pattern. Therefore, non-parallel mirror lines do not exist. We next ask if there are horizontal glide lines (perpendicular to the vertical mirror lines). The answer is no since any horizontal reflection would have to change the "arches" from being concave down to being concave up. This brings us to the final question: Are there vertical glide lines? Careful observation tells us that there are if we shift the pattern vertically half a period and then reflect it through lines like the one depicted in figure 4.4. Figure 4.5 traces the decision process we have followed and we conclude that the pattern has symmetry "cm".

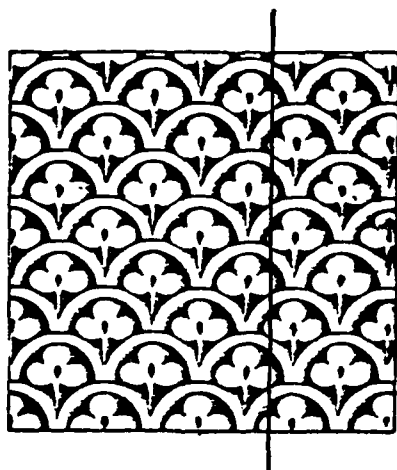


Figure 4.4

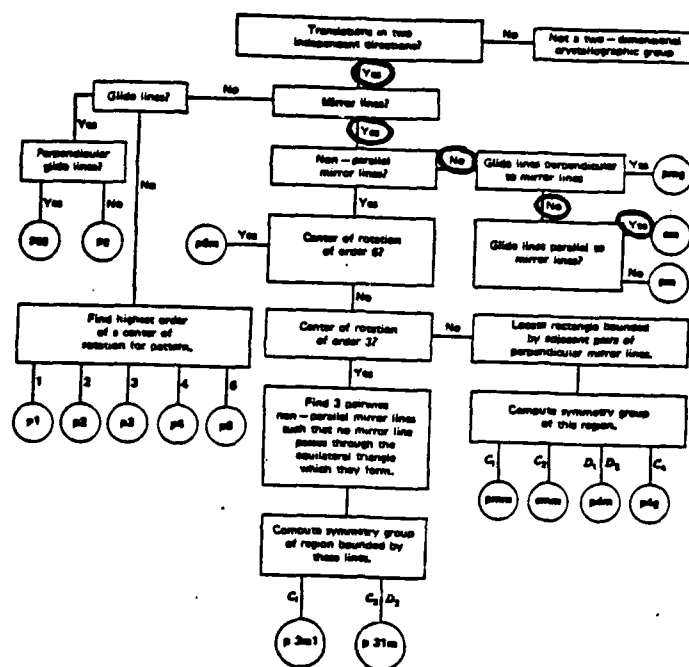
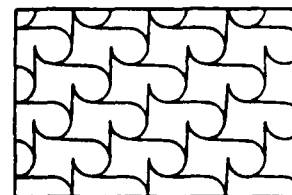
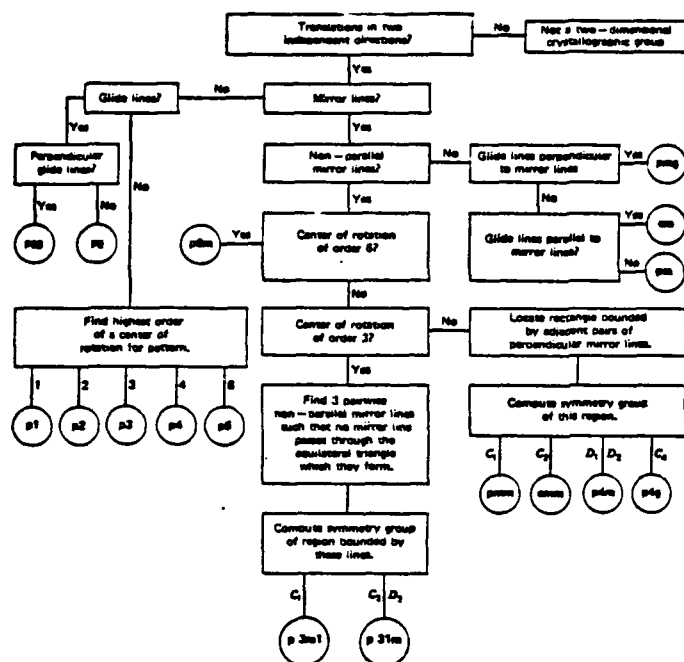
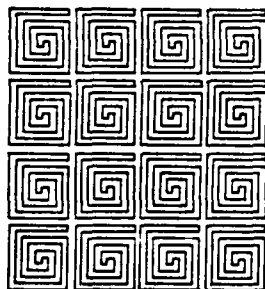


Figure 4.5

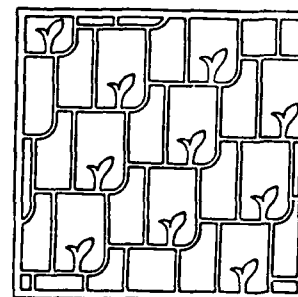
We conclude this chapter with illustrations of the 17 types of wallpaper groups. Three examples are given for each type. In every case the first example is from a paper written by George Polya in 1924. In that paper, Polya included a complete set of patterns depicting the 17 wallpaper groups. Historically, Polya's examples are important in that they were studied and copied by Escher--knowledge Escher built upon to eventually create his most brilliant designs [10]. The second and third examples are from Durbin [2] and Schattschneider [9], respectively.



1



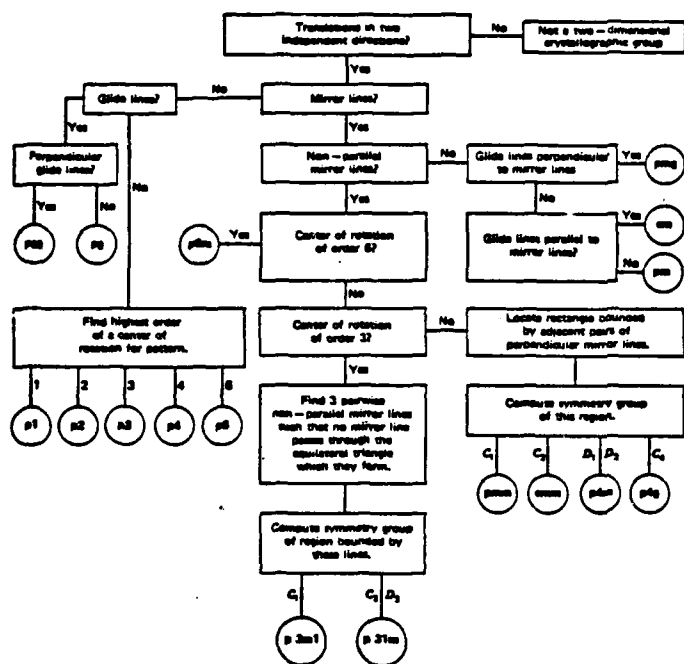
2



3

Figure 4.6

Symmetry:  $p1$



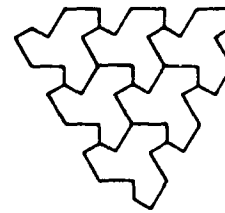
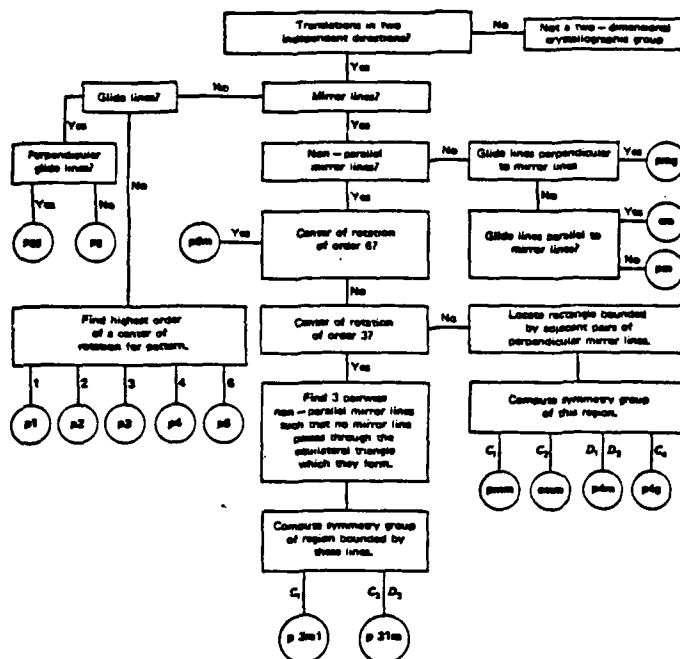
1

2

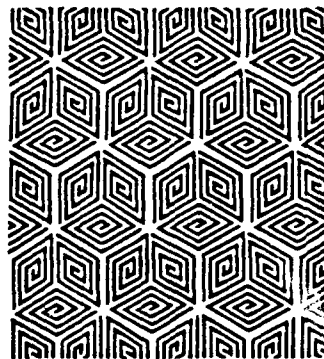
3

Figure 4.7

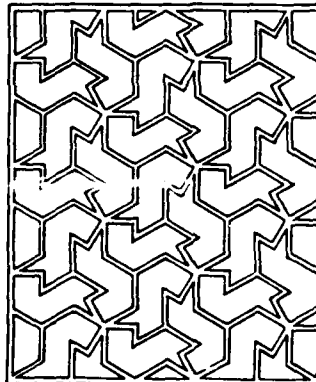
Symmetry: p2



1

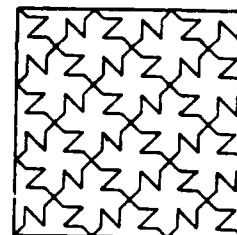
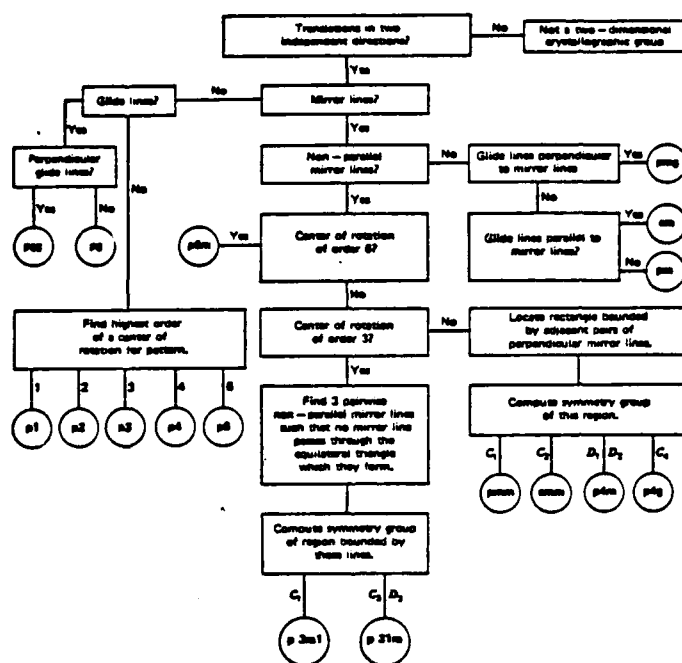


2

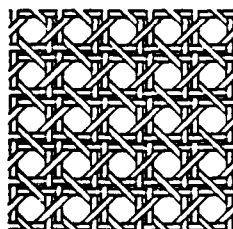


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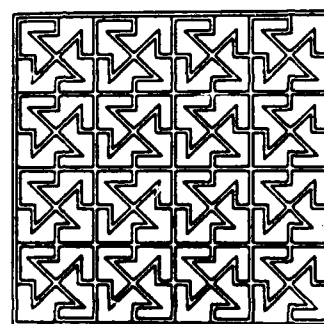
Figure 4.8  
Symmetry: p3



1



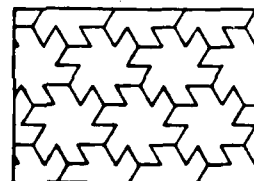
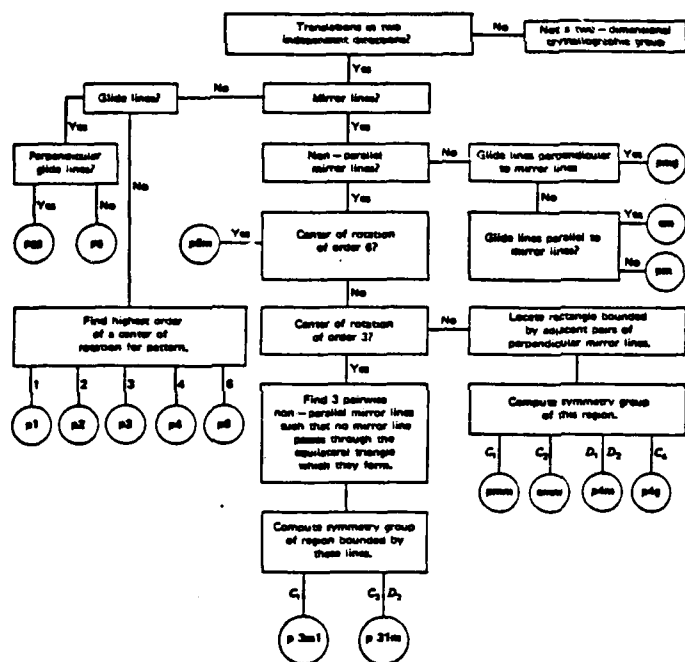
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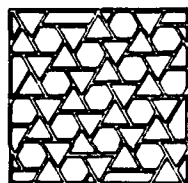
3

Figure 4.9

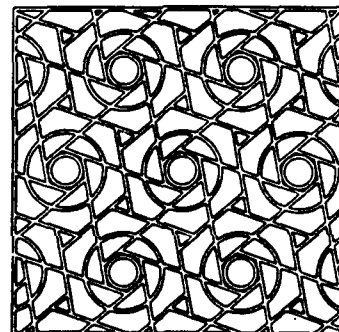
Symmetry: p4



1



2

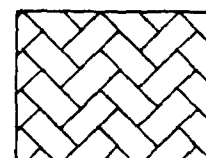
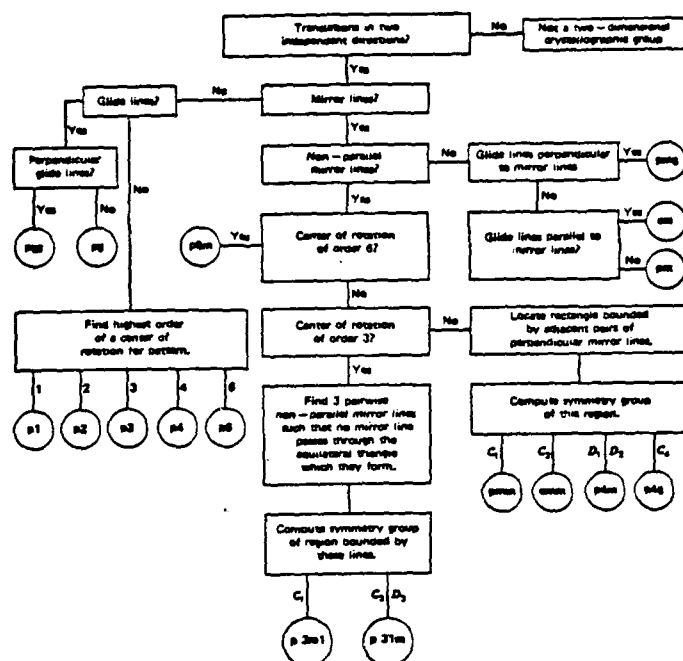


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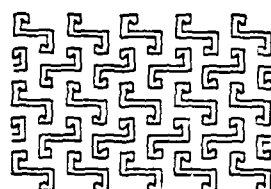
Figure 4.10

Symmetry: p6

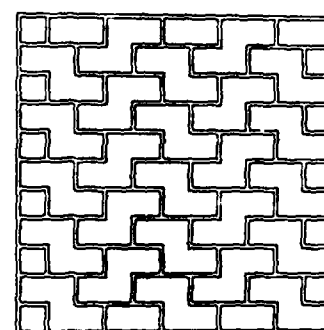




1



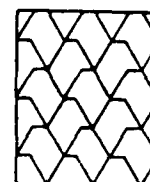
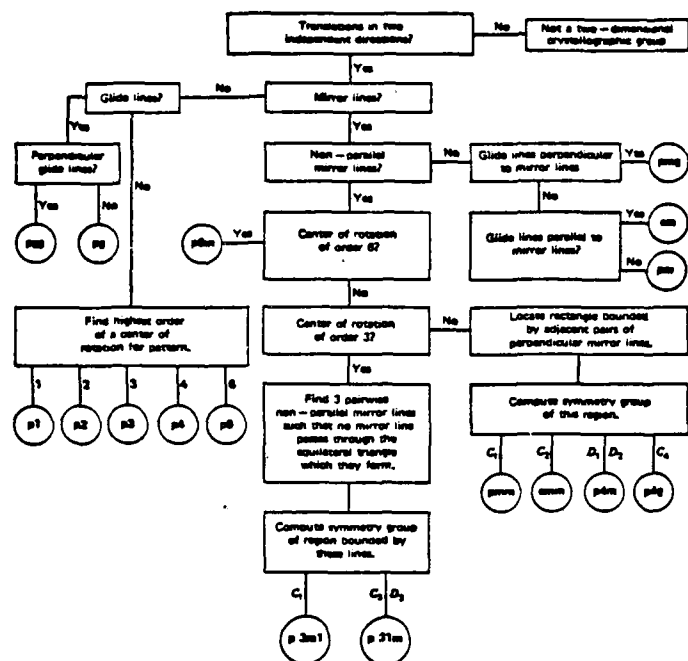
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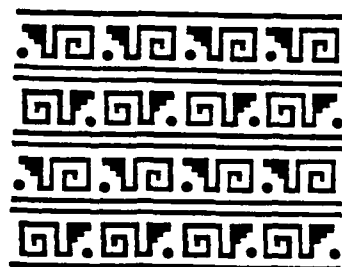
3

Figure 4.11

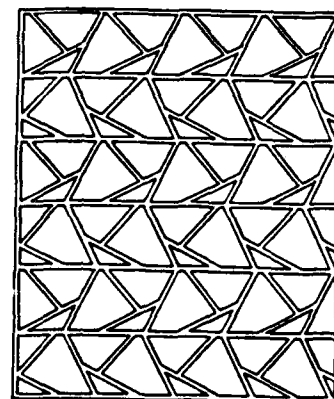
Symmetry: pgg



1



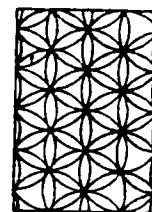
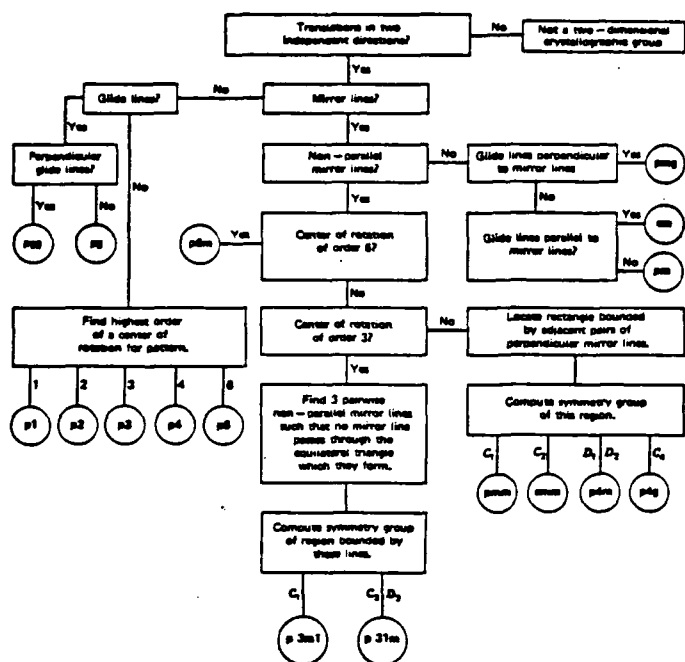
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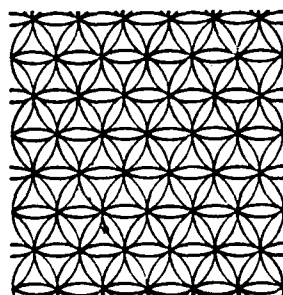
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Figure 4.12

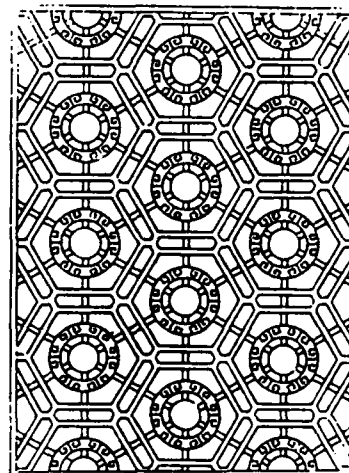
Symmetry: pg



1



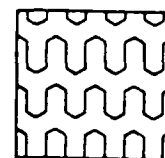
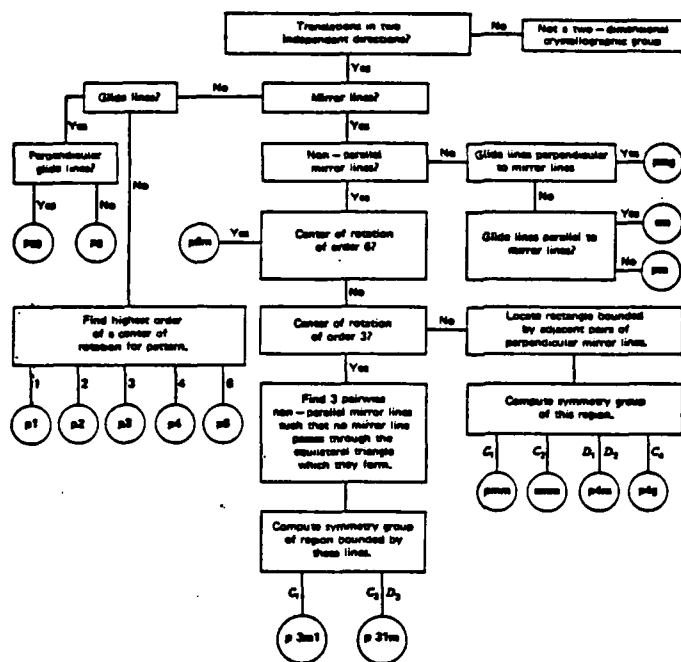
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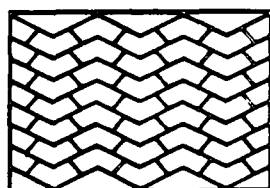
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Figure 4.13

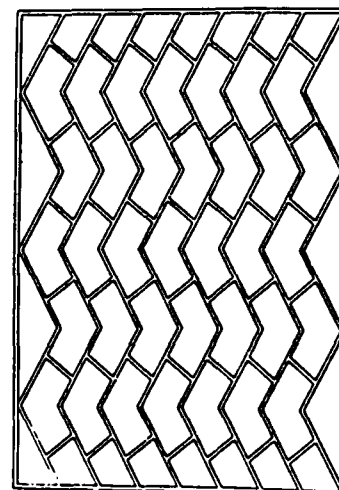
Symmetry:  $p6m$



1



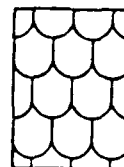
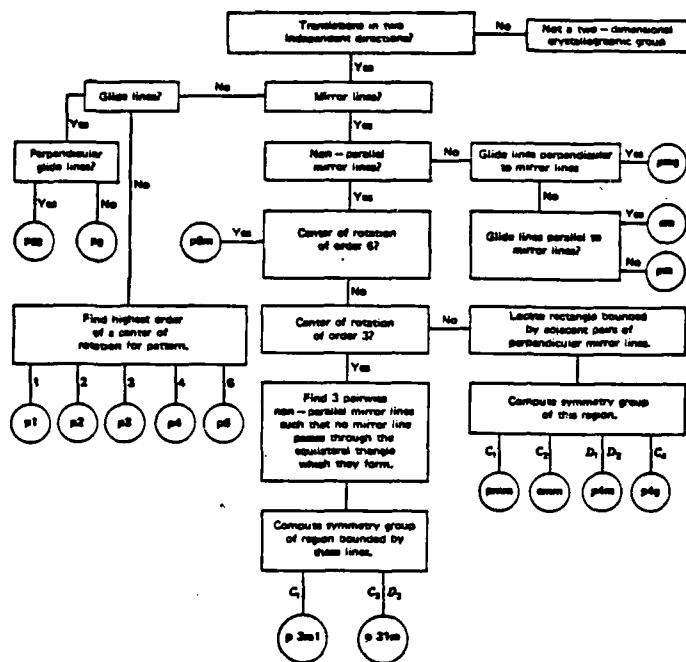
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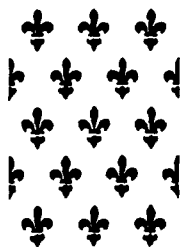
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Figure 4.14

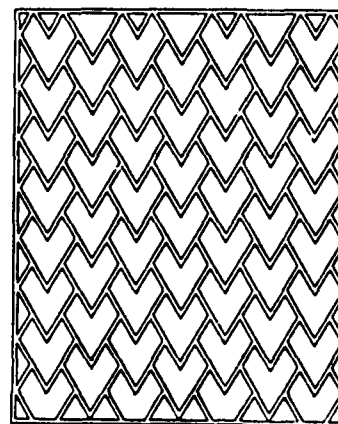
Symmetry: pmg



1



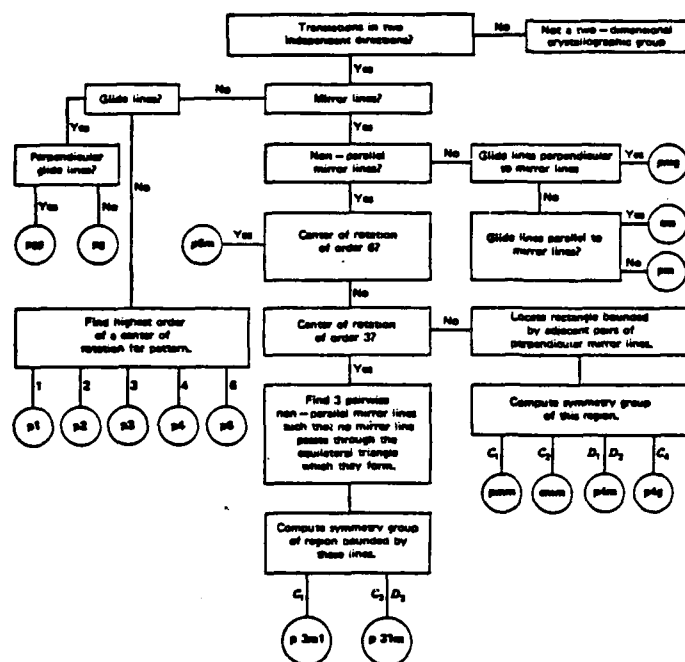
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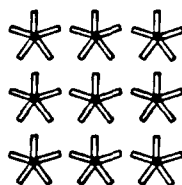
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Figure 4.15

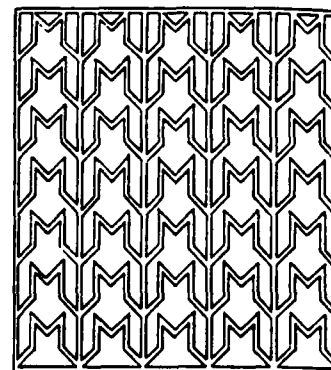
Symmetry: cm



1



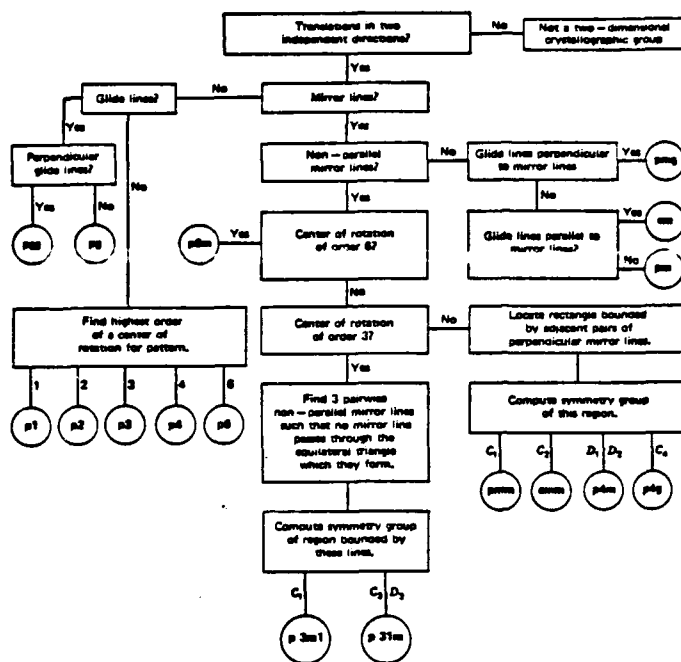
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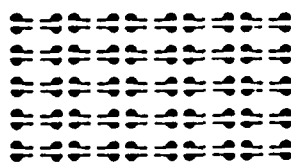
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Figure 4.16

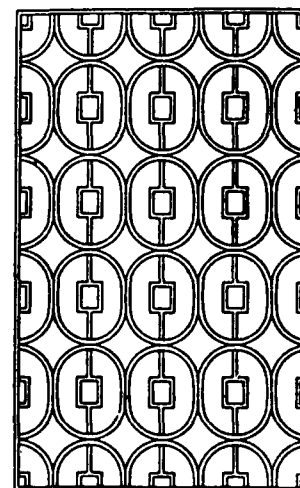
Symmetry: pm



1



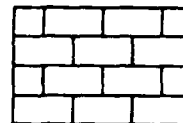
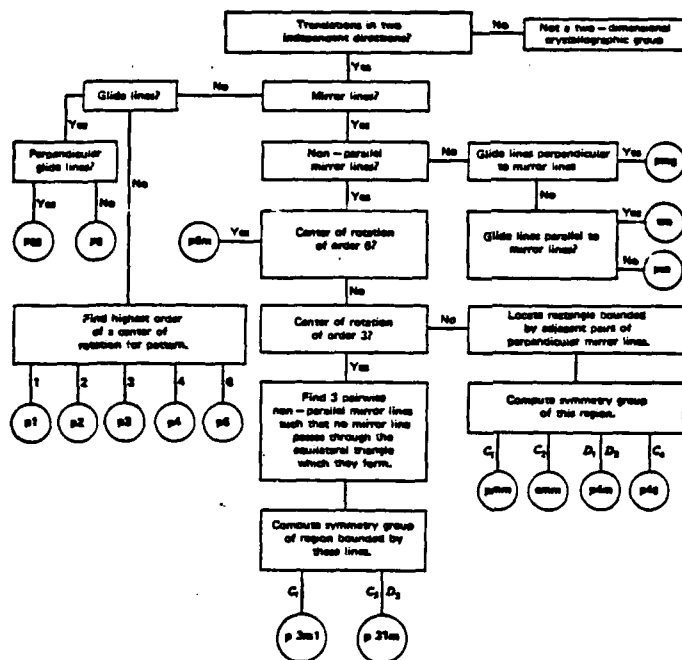
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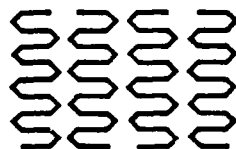
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Figure 4.17

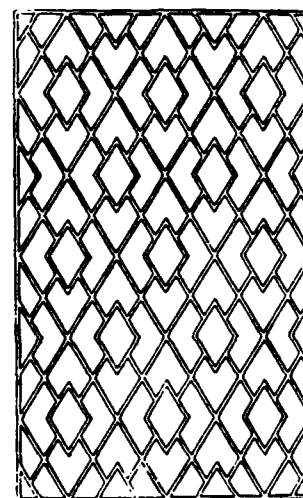
Symmetry: pmm



1



2

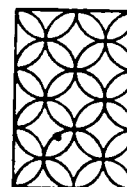
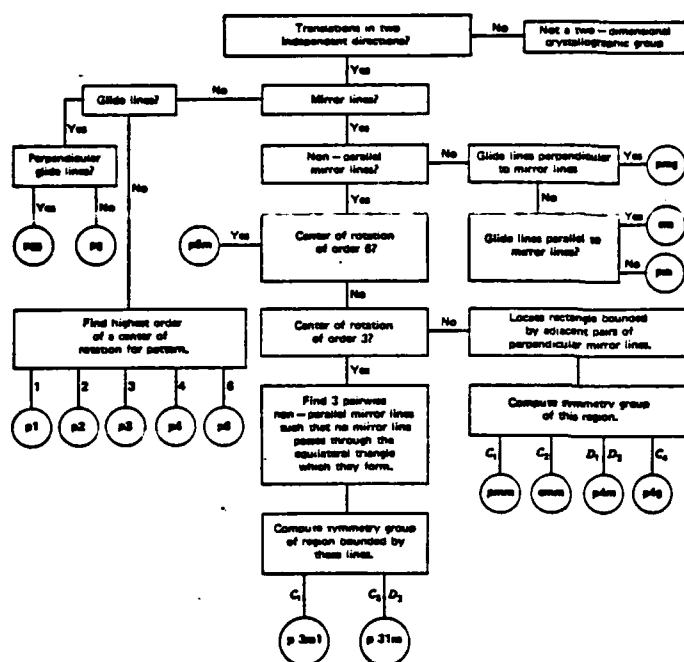


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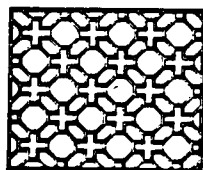
Figure 4.18

Symmetry: cmm

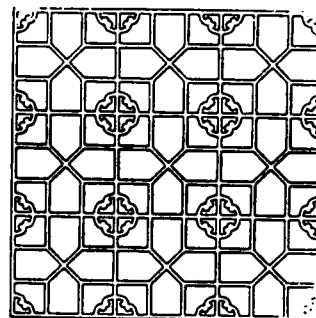




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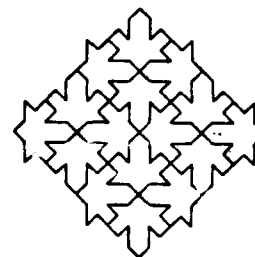
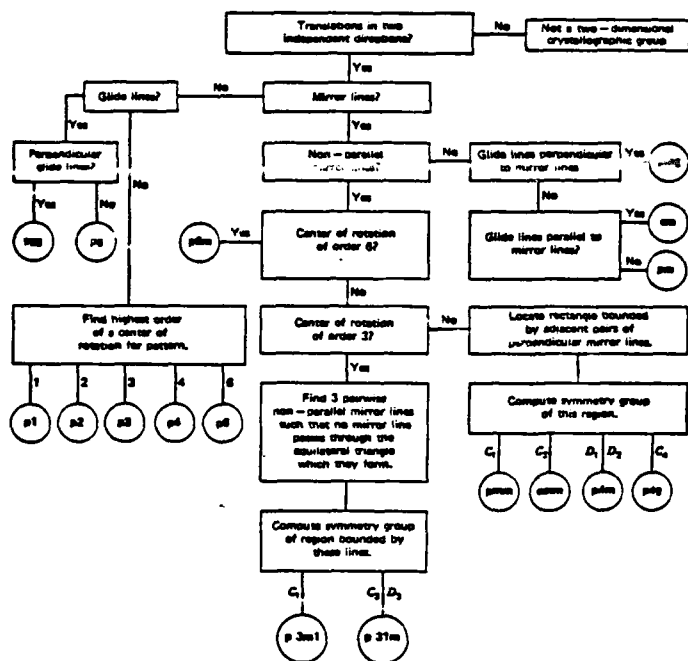
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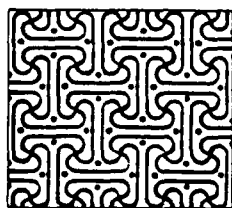
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Figure 4.19

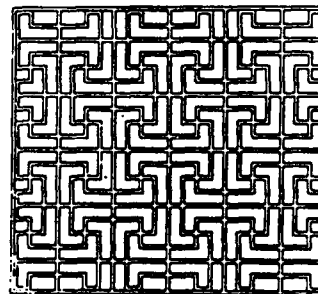
Symmetry: p4m



1

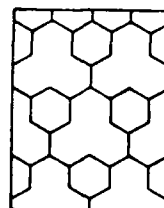
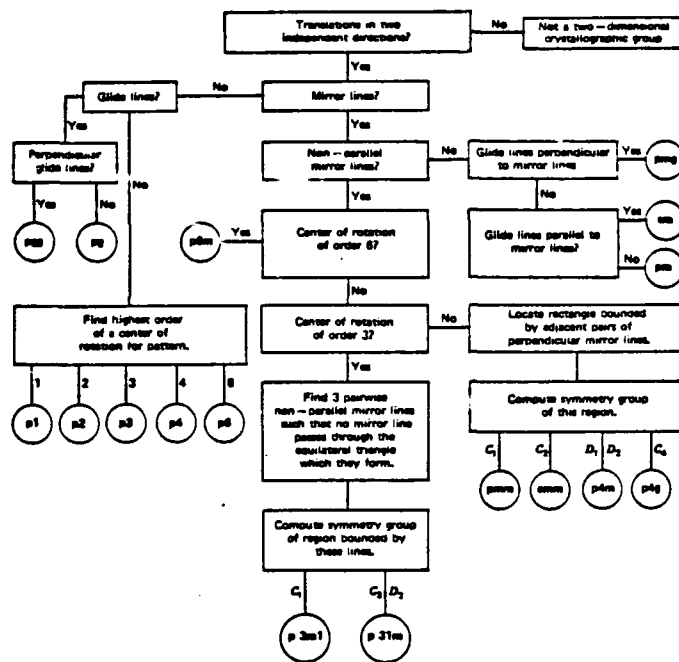


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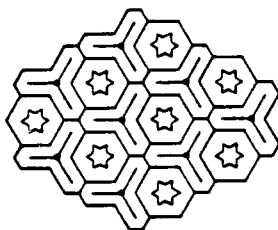


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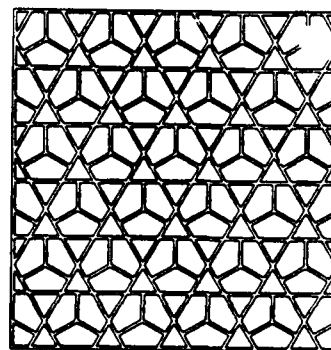
Figure 4.20  
Symmetry: p4g



1



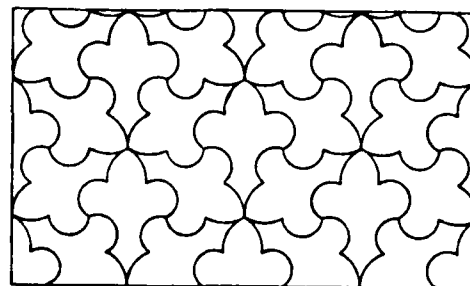
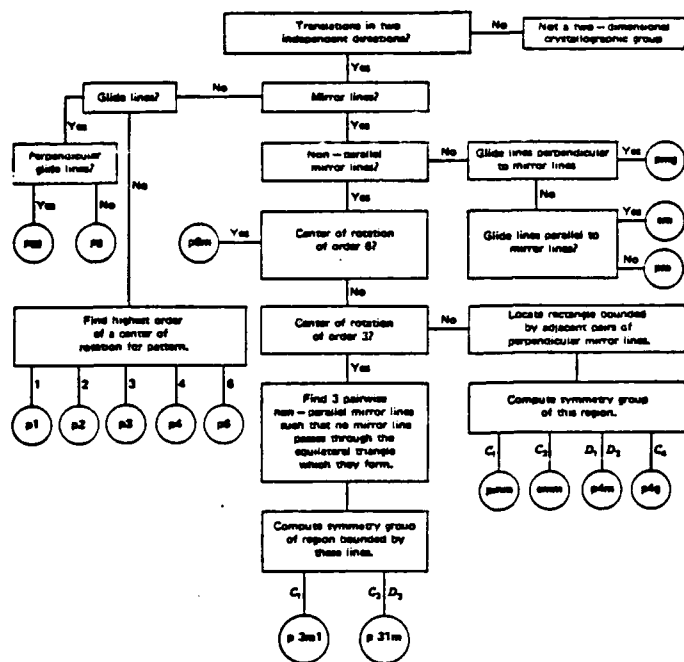
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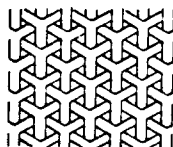
3

Figure 4.21

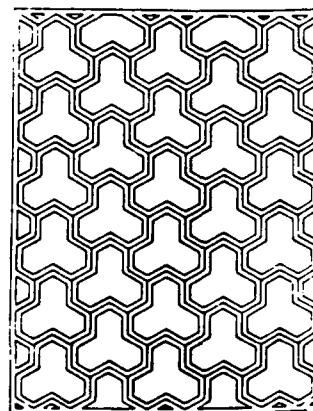
Symmetry:  $p3m1$



1



2



3

Figure 4.22  
Symmetry: p31m

## APPENDIX A

Let  $(G, \circ)$  be the group of all one-to-one, onto functions from  $R_2$  to  $R_2$  under composition and let  $H$  be those functions in  $G$  that preserve distance, that is:

$$H = \{\alpha \in G: |p - q| = |\alpha(p) - \alpha(q)| \text{ for all } p, q \in R_2\}$$

Then  $(H, \circ)$  is a subgroup of  $(G, \circ)$ .

Proof We use the following simple theorem found in every undergraduate modern algebra text: A nonempty subset  $H$  of  $G$  is a subgroup of  $G$  if and only if for all  $\alpha, \beta \in H$ , the element  $\alpha \circ \beta^{-1} \in H$ .

We start by noting that  $H$  is not empty since the identity map preserves distance. Let  $\alpha$  and  $\beta$  be any two elements in  $H$ , and let  $p$  and  $q$  be any two elements in  $R_2$ .

Notice that  $\beta \in H$  implies  $\beta^{-1} \in H$  since

$$|p - q| = |(\beta \circ \beta^{-1})(p) - (\beta \circ \beta^{-1})(q)| = |\beta(\beta^{-1}(p)) - \beta(\beta^{-1}(q))| = |\beta^{-1}(p) - \beta^{-1}(q)|.$$

What we must show is  $\alpha \circ \beta^{-1} \in H$ , that is  $\alpha \circ \beta^{-1}$  preserves distance. It follows immediately since  $\alpha \in H$  and  $\beta^{-1} \in H$  yields

$$\begin{aligned} |(\alpha \circ \beta^{-1})(p) - (\alpha \circ \beta^{-1})(q)| &= |\alpha(\beta^{-1}(p)) - \alpha(\beta^{-1}(q))| \\ &= |\beta^{-1}(p) - \beta^{-1}(q)| \\ &= |p - q| \end{aligned}$$

Q.E.D

## APPENDIX B

Let  $(H, \circ)$  be the group of all one-to-one, onto functions from  $R_2$  to  $R_2$  under composition that also preserve distance. Let  $T \subseteq R_2$ ,  $T \neq \emptyset$ , and define  $H_T = \{\alpha \in H: \alpha(T) = T\}$ . Then  $(H_T, \circ)$  is a subgroup of  $(H, \circ)$ .

Proof The identity map  $i$  is in  $H_T$  since  $i(T) = T$ . Let  $\alpha, \beta$  be any two elements in  $H_T$ . Notice that  $\beta \in T$  implies  $\beta^{-1} \in T$  since  $T = \beta^{-1}(\beta(T)) = \beta^{-1}(T)$ . We need only show  $(\alpha \circ \beta^{-1}) \in H_T$ . It follows easily since  $\alpha \in H_T$  and  $\beta^{-1} \in H_T$  yields

$$(\alpha \circ \beta^{-1})(T) = \alpha(\beta^{-1}(T)) = \alpha(T) = T.$$

Q. E. D.

### APPENDIX C

Nonzero translation: cannot be motions in the symmetry group  $H_T$  of a bounded figure.

Proof Since we are dealing with a bounded figure, pick a circle with radius  $R$  such that all of  $T$  lies inside the circle. We will assume there exists  $\alpha \in H_T$ , where  $\alpha$  is a nonzero translation with translation distance  $d > 0$ , and arrive at a contradiction. Since  $(H_T, \circ)$  is a group, it is closed under composition: in particular,  $\{\alpha^k: k \text{ a positive integer}\} \subseteq H_T$ . Note that if we apply  $\alpha^k$  to the plane we are translating every point in the plane a distance  $k \cdot d$ . For large enough  $k$ , say  $K$ , we are shifting the entire plane a distance  $K \cdot d > 2R$ , and points that start off inside the circle must end up outside the circle. But this obviously means that invariance no longer holds, that is  $\alpha^K(T) \neq T$ . We conclude  $\alpha^K \notin H_T$ , contradicting our previous statement.

Q.E.D.

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